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Reconsidering Absentmindedness

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Abstract The paper is meant to give an overview and a critical comment on the paper of Piccione and Rubinstein (1997^a) on decision problems with imperfect recall and the reactions to that paper in the same issue of Games and Economic Behavior. It investigates the logical soundness of the concepts introduced by various authors to cure the ‘absentminded driver paradox’. It poses the feasibility question ‘Is it possible (allowed) to deviate in the middle of an information set at a stochastically chosen time moment τ ?’. The paper shows that time inconsistent decision rules are, in general, better than optimal behavior rules, if the deviations are possible, and that the paradox disappears, if the deviations are not possible. The paper contains an explicit analysis of the absentminded driver story under three kinds of deviations. The analysis shows that adding to the model ‘external possibilities to solve the paradox’ causes an excavation of the information structure of the original model (the ‘absentmindedness’).

Key words: Decision problem with imperfect recall, time consistency, absentmindedness

Introduction

Emerging from absentmindedness a decision maker will usually ask himself four questions namely “Where am I?”, “What time is it?”, “What was I doing?” and “Is there something better to do?”. These are, in a nutshell, the issues addressed in Piccione and Rubinstein (1997) and the reactions to this paper in the same issue of Games and Economic Behavior. Given the imperfect recall (absentmindedness) the first question—‘in what decision node am I’—cannot be answered completely. The decision maker knows the information set he is in, but a priori nothing more. The second question makes only sense, if we have a sensible notion of *timing*. The minimal condition for such a concept is that ‘time’ increases along paths. If we know what ‘time’ is, the answers to the first two questions are quite related. So we will assume that the decision maker does not know the ‘time’ either. The third question asks if the decision maker can remember the strategy he was executing. We will assume that this is the case. We shall call this strategy the *incumbent strategy*. The last question is the main topic of all the papers on ‘absentmindedness’. If there is no better course of action, the incumbent strategy is called *time consistent*. If the decision maker cannot

remember the incumbent strategy, the last question is, of course, senseless but, even if he remembers the incumbent strategy, the question can only be answered if we know the set of alternatives the decision maker has, the ways in which he can *deviate*.

Piccione and Rubinstein argue that in a special class of decision problems with *imperfect recall*—decision problems with *absentmindedness*—the interpretation of the modelling tools in the theory of multi-step decision problems (information sets and behavior strategies) is ambiguous or incomplete and that paradoxes easily arise. The authors themselves formulated an example of a very simple decision problem with *absentmindedness* in which the use of standard techniques lead to counterintuitive or even paradoxical results. Particularly, this example—the *absentminded driver paradox*, as it is called—got a lot of attention in the subsequent reactions to the paper. A multi-step decision problem exhibits *absentmindedness* if there is a path in the decision tree that intersects at least one of the information sets more than once. This is a very special case of a decision problem with *imperfect recall*. *Imperfect recall* occurs if two different paths lead to the same information set and the paths intersect different information sets or contain different moves (in Section 1 we give a more explicit definition).

Reactions to a paradox are predictable. Some people will argue that there is no paradox and that the counter-intuitive results are the consequence of *bad reasoning*. The papers of Aumann, Hart and Perry (1997^a, 1997^b) and Gilboa (1997) belong to this category. Other people will take the problem seriously and try to formulate a more encompassing theory for multi-step decision problems. The papers of Grove and Halpern (1997), Halpern (1997), Battigalli (1997) belong to the second category.

Clearly, both approaches have their value. If the paradox is, indeed, the result of bad thinking, it is not necessary to complicate decision theory; if not, something has to be changed in the classical interpretation of decision making.

Aumann, Hart and Perry (1997^a, 1997^b) could make their point, as the analysis of Piccione and Rubinstein is—to say the least—not impeccable. The papers of the second category (Grove and Halpern (1997), Halpern (1997), Battigalli (1997)) propose many new, partially overlapping ideas to ‘solve the paradox’ and it might make sense to bring some structure in the set of proposals.

In this paper we will overview the battle field, help the wounded and bury the dead. We shall reconsider the analysis of Piccione and Rubinstein, Aumann et al. and Gilboa.

We also give our own analysis of the absentminded driver problem and connect this solution, where possible, with the concepts proposed by other authors. We will try to place the absentminded driver story in a context where the counter-intuitive results find their natural place, are no longer surprising. This is the paradox of science: although surprising, counter-intuitive results stir the intellectual potentials of the scientific community (see the many reactions to Piccione and Rubinstein), their efforts are aimed at taking away the feeling of surprise. Most people cannot stand a paradoxical situation for a long time.

Section 1 gives a synopsis of multi-step decision theory. We give special attention to concepts that are important when considering absentmindedness, namely:

- (i) the concept of *timing*,
- (ii) the concept of *deviation*,
- (iii) the interpretation of *behavior strategies*,
- (iv) the *feasibility of deviations*,
- (v) the idea of *forming beliefs and Bayesian updating*.

Section 2 restates a slightly generalized version of the absentminded driver story and gives a first analysis of the problem. We compare our analysis with the methods of Piccione and Rubinstein, Aumann et al. and Gilboa. Section 3 and 4 gives an analysis of the generalized absentminded driver story. We shall see that time inconsistent solutions are, in general, better than optimal behavior strategies, if we assume that the decision maker has a method (a device) to choose a randomized time moment to reconsider. Section 5 contains our conclusions. The most important issue in this section is the *feasibility* of a deviation.

1. A Synopsis of Multi-step Decision Theory

The subsections 1.1–1.3 introduce the basic definitions. Subsection 1.4 introduces the idea of decision problems with *timing*. In subsection 1.5 we consider two concepts of time consistency and the last two subsections are devoted to optimal strategies, forming beliefs and Bayesian updating. Apart from subsection 1.4 all the material in this section is standard. We repeat it as it gives us the opportunity to fix the notation and to emphasize some ideas that will become important in the next sections.

1.1 Multi-step decision problems.

A multi-step decision problem is usually modeled by a *decision tree*. It consists of

- a finite rooted tree (V, A, r) . (V, A) is a tree, a connected graph without cycles. $r \in V$ is a node called the *root*, denoting the starting point of the decision process. We assume that the arcs are oriented *from the root*. By the choice of this orientation we obtain a directed tree. For each node $x \in V$ we denote the set of arcs $a \in A$ starting at x by $A(x)$. For a set of nodes $X \subset V$ we write $A(X)$ for the set of arcs starting from one of the points of X .
- The set V is partitioned into three subsets Z , C and D . The set Z consists of the nodes $z \in V$ with $A(z) = \emptyset$, the terminal nodes of the tree. Here the decision process stops. The nodes in C represent the decision nodes where a ‘move of nature’ takes place. In the nodes of D the decision maker determines the direction in which the process is continued.
- For every node $c \in C$ a probability measure γ_c on $A(c)$ is given. In a point c of C a chance mechanism (with probability distribution $\gamma_c > 0$) determines the arc along which the process is continued.
- A utility function $u: Z \rightarrow \mathbb{R}$ is given. The number $u(z)$ describes the decision maker’s appreciation for reaching $z \in Z$.
- A partition \mathcal{J} of D is given. The elements X of \mathcal{J} are called *information sets*. The decision maker is assumed to be unable to distinguish the points in one information set $X \in \mathcal{J}$ but he can make a distinction between different information sets.
- For each information set $X \in \mathcal{J}$ the set $A(X)$ of arcs starting in a point of X is partitioned into *moves*. The set of all moves in X is denoted by $M(X)$. It is assumed that for every $x \in X$ and every move $m \in M(X)$ the intersection of $A(x)$ and m consists of exactly one arc. This arc is followed if the decision maker chooses move m and he happens to be in the point $x \in X$.

It is the decision maker’s task to decide, *based on the information he has at the moment of his decision*, which move to choose in each of the information sets and by doing so, he generates a lottery on the terminal nodes, the *outcome* of the decision process. We assume that the decision maker’s appreciation for a lottery on the set Z with probability distribution $\{P(z)\}_{z \in Z}$ is given by $\sum_{z \in Z} P(z) u(z)$. The decision maker is aiming at maximization of this expression (expected utility maximization).

We will call a path *from the root* to any of the nodes a *history*. So, a history consists of a non-empty finite string (of odd length) $v_0 = r, a_1, v_1, \dots, a_k, v_k$ where $v_i \in V$ and $a_i = (v_{i-1}, v_i) \in A$ is an arc of the tree. A history is a possible way along which the

process can proceed. As (V, A) is a tree, histories are uniquely determined by (and can be identified with) their last node a_k .¹

1.2 Types of decision trees

There are different types of decision trees.

(i) A decision tree exhibits *perfect information* if every information set consists of one node. Under perfect information, only the chance moves generate uncertainty and this kind of problems can be solved by *backward induction*. If at least one information set consists of more than one node, we have a decision problem with *imperfect information*.

(ii) In decision problems with imperfect information one can make a distinction between problems with *perfect* and *imperfect recall*. A decision problem exhibits *perfect recall* if for every information set $X \in \mathcal{I}$ and every pair of nodes $x, y \in X$ the following statement holds:

if the path from the root r to x intersects another (not necessarily different) information set X' and in this information set the move $m \in M(X')$ supports the path to x (meaning that only an arc in m leads to the node x) then the path from r to y intersects X' too and the same move m supports the path from r to y .

(iii) Perfect recall excludes, in particular, the possibility that one path intersects the same information set more than one time. A decision problem exhibits *absentmindedness* if some path intersects one of the information sets more often than once.

1.3 Different types of strategies

A pure strategy s is a rule that makes a choice from the available moves, one in each information set. Therefore, $s: \mathcal{J} \rightarrow M = \bigcup_{X \in \mathcal{J}} M(X)$ with $s(X) \in M(X)$ for every $X \in \mathcal{J}$. The set S of pure strategies is equal to $\prod_{X \in \mathcal{J}} M(X)$. If a pure strategy s is chosen, the chance moves generate a lottery on Z .

A *mixed strategy* is a probability measure on the set of pure strategies. A mixed strategy is supposed to be *implemented* by first randomizing over the pure strategies and to follow the selected pure strategy. This pure strategy generates (by the chance moves) a lottery on the set Z and as there is a complete order on this finite set of

¹ We do not follow the latest trend to take histories as the primitives of the model and to derive the decision tree from the set of histories. Moreover, our histories contain nodes as well as arcs and are never empty.

lotteries, it makes no sense to choose a mixed strategy. There is a pure strategy that gives an optimal outcome.

A different kind of strategy is a *behavior strategy*. A behavior strategy b assigns to each information set a strategy component b_X , a probability distribution over $M(X)$. The implementation of a behavior strategy can be done by a set of agents, one for each information set X , who represents the decision maker in the information set X and implements the behavior component b_X . These agents are dummies in the sense that they do what they are told to do. The decision maker gives the instructions and coordinates their actions (behavior components). They play an essential part in case of imperfect recall because it is hard to swallow that an—in all other respect rational—decision maker forgets some relevant information. In case of agents it is easier to accept that some agent did not get all the information or has more information than the decision maker. In information sets with absentmindedness some authors (Piccione and Rubinstein (1997^a), Gilboa (1997)), introduce *more than one agent in one and the same information set*. The terms ‘multiselves’, ‘twins’ and ‘twin-self’ are used. The ‘twin’-metaphor seems to be the favorite, as folklore says that ‘twins need not to communicate to coordinate their actions’. Thereby, twins have a mysterious communication channel that normal people do not have.

The set of behavior strategies B equals the set $\prod_{X \in \mathcal{J}} \Delta(M(X))$. Note that a *pure* behavior strategy is the same as a pure strategy: taking in each information set X one move m in $M(X)$ with probability 1 is the same as taking in each information set X the move $s(X) := m$.

In a decision problem without absentmindedness a behavior strategy b ‘can be associated’ with a *mixed strategy* σ_b . Let us do this explicitly. If $b = \{b_X\}$ is a behavior strategy, the mixed strategy σ_b puts the following weight on the pure strategy s :

$$\sigma_b(s) := \prod_{X \in \mathcal{J}} b_X(s(X)).$$

Also, in decision problems *with* absentmindedness one can associate b with σ_b but the following proposition (that makes the whole operation senseful) does not hold.

Proposition 1 *In a decision problem without absentmindedness every behavior strategy b generates the same probability distribution on the set Z (generates the same outcome) as the associated mixed strategy σ_b .*

Proof: We must prove that, for each $z \in Z$, the probability that $z \in Z$ is reached is the same under b and σ_b . Let P be the set of nodes on the path from r to z . Let C_P be the intersection $C \cap P$ and $D_P = D \cap P$. The most important observation is that *different points of D_P are from different information sets*.

To reach the node z ‘nature’ must choose the ‘right move’ in each node $c \in C_P$ and b_X must choose the ‘right move’, if $X \cap P \neq \emptyset$. So, if b is used, the chance to reach z is

$$\mathbb{P}(z, b) = \prod_{c \in C_P} \gamma_c(\text{right move in } c) * \prod_{X \cap P \neq \emptyset} b_X(\text{right move in } X \cap P).$$

We can take the product as all the chance mechanisms along P are *independent*.

If σ_b is used, we have to multiply the chance that a pure strategy s is chosen that supports the path from r to z (i.e., $s(X) = \text{right move in } X \cap P$, whenever $X \cap P \neq \emptyset$ —we call such a strategy s a *supporting strategy*) and $\prod_{c \in C_P} \gamma_c(\text{right move in } c)$.

By the definition of $\sigma_b(s)$, the chance that a supporting strategy s is chosen is equal to

$$\prod_{X \cap P \neq \emptyset} b_X(\text{right move in } X \cap P) * \prod_{X \cap P = \emptyset} b_X(s(X)).$$

To be a supporting strategy, $s(X)$ can be any move in information sets not intersecting P and must be the ‘right move in X ’ for information sets X intersecting P . If we number the information sets not intersecting P by X_1, \dots, X_q and $m_i \in M(X_i)$, we have

$$\sum_{(m_1, \dots, m_q)} \prod_{i=1}^q b_{X_i}(m_i) = \prod_{i=1}^q \left[\sum_{m_i \in M(X_i)} b_{X_i}(m_i) \right] = 1.$$

The probability that σ_b chooses a supporting strategy to reach z is

$$\prod_{X \cap P \neq \emptyset} b_X(\text{right move in } X \cap P).$$

The behavior strategy b and the mixed strategy σ_b put the same weight on every $z \in Z$. ◁

Corollary: *In decision problems without absentmindedness the set of behavior strategies can be identified with a subset of the set of mixed strategies and there is a pure (behavior) strategy that is optimal.*

If the decision problem has perfect recall we also have the converse of the corollary (Theorem of Kuhn (1953)).

For each mixed strategy s there is a behavior strategy b_s generating the same outcome (lottery on the terminal nodes) as s does.

A well known result in the same spirit as the previous proposition states:

Proposition 2. *In decision problems with perfect recall the proportions of the probabilities to reach x or x' , $\mathbb{P}(x, b) : \mathbb{P}(x', b)$ is not dependent on b , if x and x' are in the same information set X and this proportion makes sense.*

Proof: Let P and P' be the paths from the root r to x and x' , respectively. Then

$$\mathbb{P}(x, b) = \prod_{c \in P} \gamma_c(\text{right move in } c) \prod_{Y \cap P \neq \emptyset} b_Y(\text{right move in } Y \cap P).$$

For the node x' we get a similar expression:

$$\mathbb{P}(x', b) = \prod_{c' \in P'} \gamma_{c'}(\text{right move in } c') \prod_{Y' \cap P' \neq \emptyset} b_{Y'}(\text{right move in } Y' \cap P').$$

Because of perfect recall P and P' intersect the same information sets Y and for the same reason the 'right move in $Y \cap P$ and in $Y \cap P'$ are the same. This means

$$\mathbb{P}(x, b) : \mathbb{P}(x', b) = \prod_{c \in P} \gamma_c(\text{right move in } c) : \prod_{c' \in P'} \gamma_{c'}(\text{right move in } c')$$

if $\prod_{Y \cap P \neq \emptyset} b_Y(\text{right move in } Y \cap P) \neq 0$. ◁

Corollary: *In decision problems with perfect recall there is only one way to assess well founded (consistent) beliefs in information sets.*

For decision problems with absentmindedness the *implementation* of a behavior strategy is another point that must be clarified. In a decision problem without absentmindedness the behavior component b_X can be activated (the chance mechanism is put to work), when the information set X is reached. In case of absentmindedness the behavior component is activated **each time** the information set is reached. This is the opinion advocated (or at least used) in all the papers considered. Otherwise the absentminded driver paradox would disappear anyhow, because only the pure strategies 'always exit' or 'always continue' and its mixtures would remain. Note, however, that this kind of implementation weakens the statement that the same action is chosen in all nodes of an information set. But, we keep at least that the same *chance mechanism* is used in all nodes of an information set.

1.4 Timing and time functions

Another difference between behavior strategies and mixed strategies is the possibility to 'postpone' the *decision about a behavior component* for an information set 'till the information set is reached'. In the last sentence we used the time-related expressions 'postpone' and 'till the information set is reached'. To use these terms sensefully we need a notion of time or sequentiality in the decision tree. An initial notion of sequentiality is given with the tree structure (V, A, r) : *node a is before node b* if the path from r to b contains node a . We want to extend this notion to information sets. Therefore, we introduce the notion of *timing* or *time function*.

A decision tree *allows timing* if there exists a *time function* $T: V \rightarrow \mathbb{N}$ such that (i) $T(r) = 0$, (ii) T is constant in every information set and (iii) T is strictly increasing along every path.

From this definition follows immediately that decision problems with *absentmindedness* do not allow timing. In a decision problem with timing one may assume that the decisions in one information set are executed *simultaneously*.

For any decision tree the following algorithm checks the possibility of a time function. The idea of the algorithms is that a node v gets its ‘time value’ as soon as **all** predecessors (fathers) of the nodes in the same information set $X(v)$ have a time value.

We start with $T(r) = 0$, $t = 1$ and $E := Ch(r)$, the set of successors (children) of r . Next, we look for points $v \in E$ with the property $v \in C$ or $X(v)$, the information set containing v , is in E . For all nodes v with this property we define $T(v) = t$ and we replace these nodes v with $Ch(v)$. Then the value of t is increased with one unit and we repeat the operation. After a while the algorithm stops because (a) $E = \emptyset$ and $T(v)$ has been defined for all $v \in V$ or because (b) $E \neq \emptyset$ but there is no node v with the required property. In that case no time function is possible. We have to prove the last statement.

Proof: Assume the algorithm stops and $x_1 \in E$. Then there is a node $y_1 \notin E$ with $y_1 \in X(x_1)$, the information set containing x_1 . Follow the path from y_1 to the root r till a point in E is found and call this node x_2 . There is a node $y_2 \notin E$ with $y_2 \in X(x_2)$. Continuing in this way, we find after a while a repetition $x_k = x_l$ and $k < l$. Every time function would give to x_j and y_j the same value and to y_j a higher value than to x_{j+1} for $k \leq j \leq l - 1$. But since $x_k = x_l$, they have the same value. Hence, there is no time function. \triangleleft

For decision problems with perfect recall there always exists a time function.¹

Proof: (See figure 1) Suppose there is no time function. Then the algorithm stops with $E \neq \emptyset$. Take $v \in E$. The father v' of v is in an information set $X(v')$ where T has been defined already (this is the reason why $v \in E$). Let $w \in X(v)$ and $w \notin E$ (w exists because, otherwise, $X(v) \subseteq E$). If we take the path from w to the root, this path contains a node of $X(v')$ (perfect recall). Call this point w' . This is not the father of w (otherwise $w \in E$) and $T(w') = T(v')$. So, there is a node w'' , the father of w , between w' and w . The path from v to the root intersects $X(w'')$ (perfect recall) in the point, say v'' . This cannot happen between v' and v and therefore, it must have been happened before v' . Then $T(v'')$ has been defined and $T(w'')$ has not. But

¹ The following argument is no longer valid for extensive form games with perfect recall, as perfect recall only requires ‘time consistency’ for information sets of the same player and not for all information sets.

T is only defined on *complete* information sets.

◁

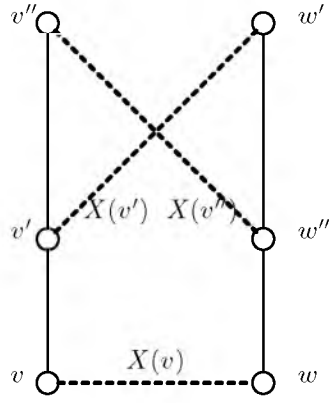


Figure 1

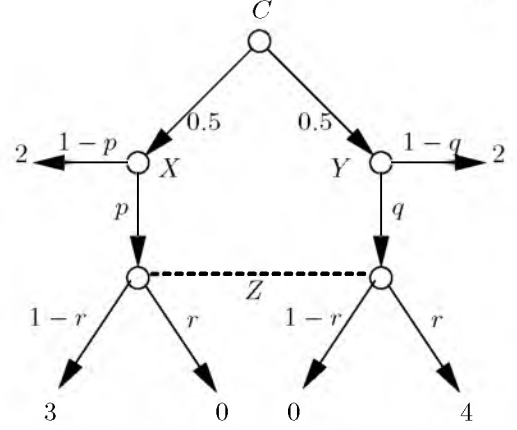


Figure 2: Imperfect recall with timing (Example 2 of P & R)

Remark If a decision problem allows timing, one can change the decision problem in an inessential way to obtain a decision problem in which all terminal paths have equal length. If the time parameter increases with more than one unit along an arc, one can introduce inessential chance nodes in between such that at every integer time moment the process is in a node. The same one can do if there are terminal nodes with a lower time parameter than the maximal time parameter $\max_{v \in V} T(v)$. Delay the payoffs till the time parameter is $\max_{v \in V} T(v)$ and put inessential chance nodes in between. If this has been done, $\{P(v, b)\}_{T(v)=t}$ is a probability distribution for every $t \in [0, \max T(v)]$ and every behavior strategy b .

Note that also some problems with imperfect recall allow timing (see Piccione and Rubinstein, example 2, see figure 2). The decision problem in Gilboa (1997) does not allow timing, although there is no absentmindedness³ (see figure 6).

Warning: If timing is not possible, the words related to the concept of time like “postpone”, “previous”, “simultaneous” and “subsequent moves” must be used very carefully. They can only be used for nodes, not for information sets.

1.5 Deviations and time consistencies

If we are dealing with a decision problem *without absentmindedness* and a behavior strategy $b = \{b_X\}_{X \in \mathcal{I}}$ is given (b_X is a probability distribution over the set $M(X)$, the set of moves in X), it is possible to speak about ‘deviating from the behavior rule in some information set X ’. To be more explicit, the components b_Y of b with $Y \neq X$

³ Even if the twins are merged.

and the chance moves γ_c defines the *chance that a node $x \in X$ is reached*: $\mathbb{P}(x, b_{-X})$. These chances do not add up to one, as there may be parallel paths (not intersecting X), followed with positive chance. These chances are **not** subjective probabilities or beliefs but uniquely determined by b_{-X} and γ_c with $c \in C$, and thereby objective probabilities

Furthermore, for every node $x \in X$ and for every move $m \in M(X)$, the behavior strategy b_{-X} and the chance moves determine a lottery over Z : $L(x, b_{-X}, m)$. The chance that a node z is reached is the product of the chances that along the path from x via m the right decisions are taken to reach z . It is well defined (i.e., independent from b_X) because, after m , the information set X is not crossed for the second time. If we denote the expected utility of $L(x, b_{-X}, m)$ by $U(x, b_{-X}, m)$, we can assign to each move $m \in M(X)$ the expected utility of the compound lottery

$$m \in M(X) \rightarrow \sum_{x \in X} \mathbb{P}(x, b_{-X}) U(x, b_{-X}, m) =: \mathbb{E}U(b_{-X}, m).$$

The decision maker is inclined to change his behavior component b_X , if it puts a positive weight on a move m that does not maximize the expected utility $\mathbb{E}U(b_{-X}, m)$.

A behavior strategy b is *time consistent in X* if the behavior component puts only positive weights on moves with maximal $\mathbb{E}U(b_{-X}, m)$. If this is true for every information set X , we call the behavior strategy b *time consistent in each information set*.

Note that we are talking about changing the behavior in just one information set.

Another kind of deviation allows the decision maker to change the current decision and all ‘future’ decisions. Here we get a different (stronger) type of time consistency. The use of the time-related expression ‘future decisions’ makes clear that the decision maker should know what are the ‘future decisions’. This is the place where ‘timing’ becomes an important property. If no timing is possible, it makes no sense to speak about ‘future decisions’ or ‘future information sets’.

In a decision problem *with timing* an information set Y *follows* the information set X if there is a path from the root to at least one node of Y that intersects X . It is not necessary that every node in Y has this property. So, it is allowed that some point(s) of Y can be reached from r without intersecting X . If the decision problem exhibits perfect recall, then the stronger condition—every path from r to Y intersects X —follows automatically. We denote the collection consisting of X and all information sets *following* X by \mathcal{J}_X . Now the decision maker can delete the components b_Y with

$Y \in \mathcal{J}_X$ and take new decisions in \mathcal{J}_X . We denote the behavior strategy with the decisions in \mathcal{J}_X deleted by $b_{-\mathcal{J}_X}$.

For all nodes v in X and maybe also for some nodes in information sets that follow X the chance to reach v is not dependent on $b_{\mathcal{J}_X}$. This is the case if only the last point of the path from r to v is in X or an information set following X . Then, $\mathbb{P}(v, b) = \mathbb{P}(v, b_{-\mathcal{J}_X})$. So, we may consider the new decision problem consisting of the information set X and the information sets that follow X . We add a chance node c_0 , connected with all nodes v for which $\mathbb{P}(v, b_{-\mathcal{J}_X})$ is well-defined and define γ_{c_0} by $\gamma_{c_0}(c_0 \rightarrow v) := \mathbb{P}(v, b_{-\mathcal{J}_X})$. We call the new decision problem the *subproblem from X downward, induced by $b_{-\mathcal{J}_X}$* . Note that the sum of the probabilities of γ_{c_0} might be less than one. If the sum is not zero, one can, if one wishes,⁴ multiply the probabilities with a positive factor to make the sum one. It makes no difference for the (optimal) decisions in the subproblem but ‘updating’ has the disadvantage that the expected utility levels of the subproblem can no longer be compared with the utility level(s) outside X . The contributions from X downward are overweighted.

A behavior strategy b is *time consistent in downward direction from X on* if $b_{\mathcal{J}_X}$ is optimal in the subproblem induced by $b_{-\mathcal{J}_X}$. A behavior strategy b is called *time consistent in downward direction* if it is time consistent in downward direction from every information set X on.

These two kinds of deviations make sense but changing the behavior rule b everywhere but keeping the beliefs as if b is used, does not make sense (see e.g., the definition of *time consistency* in Piccione and Rubinstein and Battigalli (1997)).

If X is an information set *with absentmindedness*, $\mathbb{P}(x, b_{-X})$ is only defined in the ‘upper frontier’ of X (the points $x \in X$ where some path from the root r enters X for the first time) and $\mathbb{EU}(x, m, b_{-X})$ can only be defined in the ‘lower frontier’ of X (the points from where no path to a (reachable) terminal node does intersect X anymore). The decision maker cannot distinguish the points in the upper frontier, the lower frontier and the points in-between. Nevertheless the upper frontier is a crucial concept in Halpern’s Game Tree Time Consistency (Halpern (1997), p. 79). To define time consistency in decision trees with absentmindedness one must assume that the behavior component b_X is followed during a part of the decision process inside X . So, the deviation must take place *inside* the information set X . This is a completely different kind of deviation and time consistency. In one information set there are now

⁴ Some people become nervous, when chances do not add up to one.

two or more behavior components. If we allow such deviations, in different nodes of an information set different *chance mechanisms* are used. Here the *feasibility problem* arises: ‘Is it in accordance with the spirit of the model to deviate in the middle of an information set’. The least that can be said is that we are leaving the set of behavior strategies, if we allow such deviations!

1.6 Optimal decisions

As said before it is the task of a decision maker to find a decision rule that maximizes the expected utility of the outcome within the set of *feasible decision rules*. For the moment we assume that the set B of behavior strategies is the set of feasible decision rules.

In general an optimal behavior strategy can be *found* by a combination of *forward* and *backward induction*. Forward induction (reasoning about the past) enables an assessment of probabilities in each of the information sets; by backward induction (reasoning about the future) the decision maker can make a well-founded choice from the available moves in an information set. In decision problems with perfect recall the first part is not dependent on the past (see Proposition 2) and a backward induction reasoning solves the problem. In a decision problem with imperfect information but with timing the probabilities are uniquely determined by *previous* chance moves and *previous* decisions, made by the decision maker i.e., by the past.

If the decision maker has, in an information set, these well-founded beliefs about the decision node where he is, he has to make a choice from the available moves. To be able to do so, he should know the consequences of each of the possible moves. Here the backward induction reasoning starts. If (some of) the possible moves lead to a new decision node, the decision maker must know what he will do in each of these nodes. *He must anticipate his future decisions*. If, however, the moves in the present information set also lead to information sets with imperfect information, the *usual circularity* appears: to decide in the present information set requires knowledge about future decisions and to anticipate future decisions the present decision must be known (to assess well-founded beliefs). In problems without absentmindedness the traditional way-out is to take at the same time (a) a *behavior strategy* $b = \{b_X\}_{X \in \mathcal{J}}$, i.e., a randomizing strategy for each of the information sets X , and (ii) a probability vector in each information set X : $\mu = \{\mu_X\}_{X \in \mathcal{J}}$. The behavior strategy b and the belief vectors μ are called *consistent* (fit together) if

μ_X is proportional to the chance vector $\{P(x, b_{-X})\}_{x \in X}$ and

b_X maximizes $a_X \in \Delta(M(X)) \rightarrow \sum_{x \in X} \mu(x) \sum_{m \in M(X)} a_X(m) \mathbb{E}U(x, b_{-X}, m)$.

The behavior strategy is a fixed point in the process of *reconsidering*. This is the concept of time-consistency in every information set, as introduced in subsection 1.5.

Remark: The ‘belief’ μ_X is the formal Bayesian update of the probabilities to ‘reach’ the node x if b_{-X} is applied, $\{\mathbb{P}(x, b_{-X})\}_{x \in X}$. Therefore, $\mu_X(x)$ admits the interpretation ‘the probability that x is reached under the condition that X is reached and b is applied’ if and only if the events ‘ x is reached under b_{-X} ’ are disjoint. This is the case if and only if the information set X exhibits no absentmindedness. The transition from this interpretation of μ_X to the interpretation ‘being in x under the condition of being in X ’ is based on the frequentistic interpretation of probabilities. Repeat the stochastic process b_{-X} several times and count how often you are in x if you finish in X . In case of absentmindedness the Bayesian update of $\{\mathbb{P}(x, b)\}_{x \in X}$ is meaningless and identifying the ‘formal’ update of $\{\mathbb{P}(x, b)\}_{x \in X}$ with $\{\mu_X(x)\}$ needs justification. The frequentistic justification will not suffice, as the events ‘being in x ’ are not defined in a dynamical context and the events ‘reaching x ’ are not disjoint. Updating makes them disjoint in an artificial way. Piccione and Rubinstein (1997) and Halpern (1997) use this argument, nevertheless; the latter author with some hesitation. In Figure 3 e.g., the probabilities of ‘reaching b ’, ‘reaching c ’ and ‘reaching e ’ are 0.5 but the events ‘reaching c ’ and ‘reaching e ’ are the same. If we repeat the process 100.000 times and *every time that we are in $X = \{b, c, e\}$* we look in which point we are, we may find e.g., 48,985 hits for b . Then we find necessarily 51,015 hits for c and for e . It is simply wrong to conclude that, if the process is in X , the probability to be in b is $\frac{1}{3}$. To see why, assume that the chance move at a is implemented by taking a number n from $[1, \dots, 100]$ at random (equal probabilities) and going to b if n is odd and to c if n is even. In b we take the same number and go to d if n^2 is odd (always the case and therefore an innocent action) and in c we go to e if n^2 is even. “Bayesian updating” would give the silly result that the number n is odd with chance $\frac{1}{3}$, is even with chance $\frac{1}{3}$ and that n as well as n^2 are even with chance $\frac{1}{3}$.

When so many intelligent authors assign to the Bayesian update of $\{\mathbb{P}(x, b_{-X})\}_{x \in X}$ the meaning of a subjective probability, there must be some truth in it. So, let us start again.

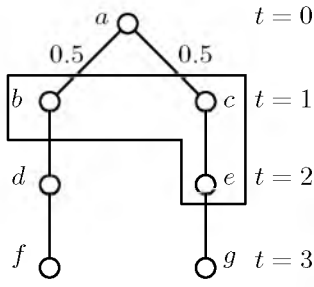


Figure 3: Wrong justification

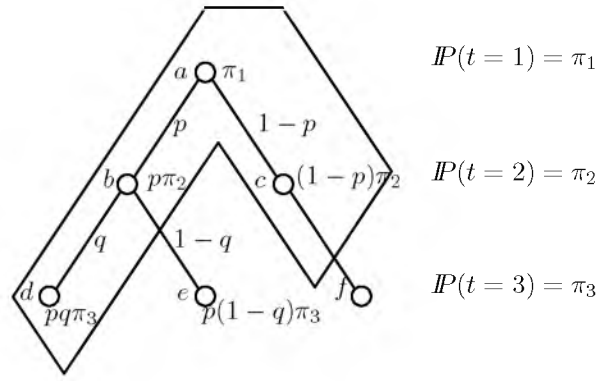


Figure 4: Right justification for updating

A fixed Bayesian strategy b generates a dynamic process, also called b . It says at time $t = 0$ the process is in the root, then the chance mechanism (given by b_r or the chance move γ_r) is executed and the next moment the process is in node such-and-such with chance so-and-so and so on. In this dynamical context there is no event ‘being in x ’ or ‘being in X ’ but only events like ‘being in x (or X) at time t ’ or ‘being in x at any time’ (‘reaching x ’) are well defined. So, the right events are $[(x, t); b]$, “the process b is at time t in node x ”. Furthermore, for every node x there is one time moment $t = t(x)$ such that $[(x, t); b]$ is logically possible (not excluded by the decision tree). The set X must be replaced by the set of logically possible states $X' := \{(x, t(x)) : x \in X\}$. The events $[x, t]$ are automatically disjoint. What Grove and Halpern (1997) call the event ‘at x ’ seems to be the event $[x, t(x)]$

If we assume that the dynamical process is inspected at time t , a moment in time chosen by a chance mechanism, independent of the decisions in b , the probability that the process is in x at time t under the condition that $(x, t) \in X'$ equals the Bayesian update of $\{IP(x, b)IP(t = t(x))\}_{x \in X}$. $IP(x, b)$ is the probability that ‘ x is reached’ and $IP(t = t(x))$ is the probability that we inspect the process at time $t = t(x)$. If, moreover, the probabilities $IP(t = t(x))$ are the same for all values of $t(x)$ with $x \in X$, the Bayesian update of $\{IP(x, b)IP(t = t(x))\}_{x \in X}$ and of $\{IP(x, b)\}_{x \in X}$ are the same. Therefore, under the assumption that an external process chooses a time moment t in the set of possible time moments $\{t(x) : x \in X\}$ and each of these time moments has the same probability to be chosen, then the ‘formal Bayesian update’ of $\{IP(x, b)\}_{x \in X}$ are the conditional probabilities of the event $(x, t(x))$ under the condition that one of the logically possible states $(x, t(x))$ with $x \in X$ is occurring.

The following example may clarify the situation (see figure 4). Suppose a dynamic process b starts at point a , and goes from there with probabilities p and $1 - p$ to the

points b or c , respectively. From b the process goes with probabilities q and $1 - q$ to d or e respectively. Let X be the set $\{a, b, c, d\}$. Then $t(a) = 1$, $t(b) = t(c) = 2$ and $t(d) = t(e) = 3$. Let us assume that the time moment t is chosen to be $t = i$ with probability π_i for $i = 1, 2$ and 3 . The events $(v, t(v))$ have the probabilities π_1 , $\pi_2 p$, $\pi_2 (1 - p)$, $\pi_3 pq$ and $\pi_3 p(1 - q)$ for $v = a, b, c, d, e$. The probabilities do not add up to 1, as we did not tell what happened at $t = 3$ if the process was in c at $t = 2$. Let us assume that it leaves the set X with certainty. If (and only if) $\pi_1 = \pi_2 = \pi_3$, the Bayesian update gives the chances

$$\frac{1}{pq + 2}, \quad \frac{p}{pq + 2}, \quad \frac{1 - p}{pq + 2} \quad \text{and} \quad \frac{pq}{pq + 2}.$$

This is equal to the formal update of $\{P(\text{reach } v, b)\}_{v \in X}$.

For decision problems with perfect recall one has the *surprising* result that ‘fitting together’ is enough for ‘optimality’. It is quite conceivable that the best continuation of ‘doing the wrong things (behaving suboptimally)’ is ‘doing the wrong things’ and that starting with ‘doing the wrong things’ is the best thing you can do if you will do it in the future. So, we rephrase the theorem of Hendon et al. (1996) for decision problems:

Theorem 3 *If a decision problem has perfect recall, a behavior strategy is optimal if and only if it is time consistent in every information set (also in information sets reached with probability zero).*

The theorem says that in case of perfect recall the agents can decide *independently* to reach an optimal outcome. If the agents play a sequential equilibrium in the ‘agent normal form game’, optimality is guaranteed.

For decision problems with imperfect recall and especially for problems with absent-mindedness the situation changes completely. Both steps in the decision process (the assessment of probabilities and the choice of the actions) must be reconsidered. But, before we come to the decision making process an all important question must be posed, namely the question about the *feasibility* of certain decision rules. “What decision rules” is the question “can be implemented without upsetting the underlying idea of absentmindedness?” Also, in problems with perfect recall we meet this problem: it is e.g. not allowed (not possible) to make decisions, conditional to being in a special node of an information set. The decision cannot depend on information (the model says that) the decision maker does not have. The advice “to sell if stock prices will go down and to buy if they will go up” may be an optimal advice, it is certainly not

a **feasible** advice. From our analysis of problems with perfect recall we found that the optimal expected utility can be obtained by following a deterministic behavior strategy and the feasibility of this kind of decision rules is beyond any doubt and, in fact, never questioned.

In decision problems with imperfect recall the feasibility problem is a serious one, as some authors are inclined to ‘cheat a bit with the absentmindedness’. Halpern e.g., introduces an external process that chooses a node in an information set as the only place where the decision maker reconsiders his strategy (see the definition of PR-time consistency). This is an extension of the model that Aumann et al.(1997) rejects. In Grove and Halpern (1997) the ‘expected utility paradox’ (p. 53, formula (1)) is based on the assumption that the decision maker takes *different* decisions (or at least different considerations) in *different* points of one information set.

1.7 Forming beliefs and Bayesian updating.

In Piccione and Rubinstein the abilities of a decision maker in an information set are summarized as follows

he cannot distinguish the points in the set he has reached. He can, however, make inferences and form a belief.

And later

he forms beliefs about the histories which led to it (the information set). These beliefs

are the basis for his considerations.

The authors are remarkably silent about this ability ‘to form beliefs’⁵ and this hurts because the beliefs are important as ‘basis for his considerations’ (choice of actions, we suppose). In the analysis of the absentminded driver story (see next section) the ‘beliefs to be at the first or the second crossing’ are just dropped. They come from the blue sky but it is ‘the basis for further considerations’. If the beliefs are wrong, the following considerations might be wrong. As long as the decision maker takes the right decisions his beliefs and considerations are immaterial. His beliefs are only important in as far as they *explain why the decision maker took the decisions he has taken* or *as a mean to find optimal decisions*. A decision maker is not obliged to form beliefs but if he does, he should tell where his beliefs come from. He must have *well founded beliefs*. Aumann et al. find the foundations for their beliefs in two observations: (i) the stochastic time moment \mathbf{t} has value 1 or 2, each with probability 0.5 (this fact

⁵ and most authors are

is used in the appendix to derive the conditional expected utility formula $h(p, q)$ and (ii) up to the moment \mathbf{t} the strategy q is played. Based on these two observations the beliefs are no longer beliefs but objective probabilities. One can argue that in decision problems (in contrast with extensive form games) ‘beliefs’ do not occur or make no sense. They are chances (objective probabilities) or they are based on nothing. If one does not know, one should not pretend to know.

Bayesian updating is an other subject where problems can arise (see the remark in subsection 1.6). It is a method to get new probabilities from old probabilities and the theory says that it only gives meaningful results if the events that occur are disjoint. Of course, the arithmetical exercise can be performed in more situations (as long as dividing by zero is avoided) but the results lose their meaning. Moreover, it is quite often not necessary to update chances, even if it is allowed. A drawback of updating is that expected utilities after updating cannot be compared with expected utilities before updating. This is one of the mistakes Piccione and Rubinstein make.

2. The Absentminded Driver.

Let us first repeat a slightly generalized version of the *absentminded driver* story.

Example 1. (the absentminded driver; see Piccione and Rubinstein (1997) for $n = 2$ and Aumann et al. (1997^a) for $n = 3$)

After a visit to a bar a decision maker has to make an attempt to get home. To reach his home he (knows that he) must follow the highway and to take the (say) p -th exit. If he takes a different exit or no exit at all, he comes into a situation that has a lower utility than by ‘taking the p -th exit’. The highway has n crossings, called C_1 , C_2 up to C_n and taking the j -th exit leads to the terminal node Z_j . Never taking an exit leads to $Z_{n+1} = M$. The decision maker knows that, if he is on the highway, he cannot distinguish the crossings C_j . Or to say it technically, all crossings together form one information set. The terminal nodes Z_j give the decision maker a utility of u_j . Accordingly, at each decision node C_j ($j \geq 1$) the decision maker has two options H (to follow the highway) and E (to choose the present exit). In the bar (B) he has only the option H . In Piccione and Rubinstein (1997) the numbers are $n = 2$, $p = 2$ and $u = (u_1, u_2, u_3) = (0, 4, 1)$. In Aumann et al. (1997^a) the numbers are $n = 3$, $p = 3$ and the utilities are $u = (7, 0, 22, 2)$ (see figure 5). We assume that neither Z_1 nor Z_{n+1} has the highest utility (otherwise the problem is trivial).

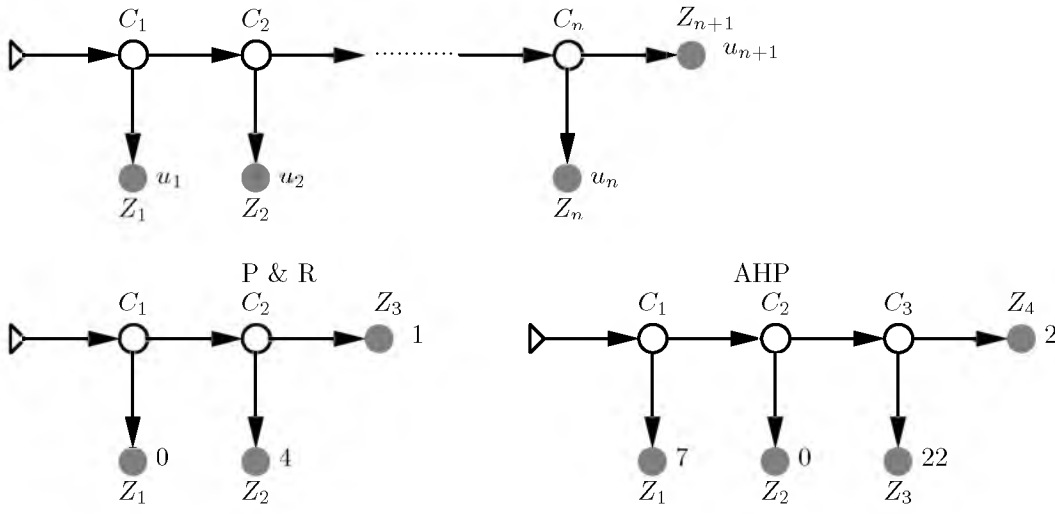


Figure 5: The absentminded driver.

As usual, the question is what course of action the decision maker should choose to maximize his (expected) utility. But, a question that must be answered before is “What course of action *can* be chosen” or “What are the *feasible* action plans, compatible with the (spirit of the) model?”. Piccione and Rubinstein start with only two strategies H^n and E^n : always following the highway and ending up in Z_{n+1} or (always) leaving the highway which ends in the terminal node Z_1 . So, the maximum of u_1 and u_{n+1} can be obtained by a deterministic strategy. If also *behavior strategies* are allowed, the set of feasible actions is extended to the 1-dimensional set consisting of the strategies $[p : 1 - p]^n$ with $0 \leq p \leq 1$: at each crossings the option H is chosen with chance p and the option E with chance $1 - p$. It is not necessary to distinguish the different crossings to implement this strategy. But the decision maker must be equipped with a ‘ $[p : 1 - p]$ -randomizer’ that performs its duty at each crossing. Thereby, the randomizer is also, at some extent, the decision makers’ memory. The decision maker *knows at each crossing the behavior strategy he is performing: he remembers his strategy*. In the subsequent discussion we will assume that the strategy is implemented by the following device: the decision maker hires a taxi (anyhow better, when leaving a bar), installs a ‘ $[p : 1 - p]$ -randomizer’ on the dashboard and instructs the driver to leave the highway as soon as the display shows EXIT. This is a less futuristic version of the automatic car of Aumann et al. (1997). The outcome of this strategy is a lottery:

$$[(1-p) : Z_1] \oplus [p(1-p) : Z_2] \oplus \cdots \oplus [p^{n-1}(1-p) : Z_n] \oplus [p^n : Z_{n+1}].$$

Therefore, maximizing the function $F: p \in [0, 1] \rightarrow \sum_{i=1}^n p^{i-1}(1-p)u_1 + p^n u_{n+1}$

solves the problem. In Piccione and Rubinstein's example (P & R from now on) $p^* = \frac{2}{3}$ is the unique optimal solution, yielding an expected utility of $\frac{4}{3}$. In Aumann et al. (AHP), $p^* = 0$ is the optimal solution with utility level 7. In the first case the extension of the set of feasible action plans gives an increase of the maximal expected payoff. One could call this the *Small Absentminded Driver Paradox*, as this phenomenon does not occur in problems without absentmindedness (see the Corollary of Proposition 1 and the Theorem of Isbell (1957) as quoted in Piccione and Rubinstein, Proposition 3).

So far, Piccione and Rubinstein and the reactions by Grove and Halpern, Lipman, Halpern, Battigalli, Gilboa and Aumann, Hart and Perry (all in the same (1997)-volume of Games and Economic Behavior) agree. The next extension of the set of feasible strategies, introduced by Piccione and Rubinstein and by which they construct their *Absentminded Driver Paradox* is the possibility to *reconsider the chosen behavior strategy at some stochastically chosen time moment* τ ($\tau = 1, 2, \dots, n$). This is our interpretation of the ambiguous sentence in P & R:

Reaching the intersection the driver will form beliefs.

The decision maker follows a behavior strategy $[p : 1-p]^n$ up to a stochastically chosen time moment τ , at time τ he 'reconsiders' his strategy choice and *is allowed to deviate*. Only the last part of the sentence describes the extension of the strategy space. Only reconsidering without deviating may make you *feel sorry* (cf. Aumann et al. (1997^a)) but does not change your expected utility. To be a *complete description of the set of new strategies* we still miss the part that describes the kind of deviations that are allowed. We return to this issue in due time.

Time inconsistency occurs, when the decision maker has reasons to change his course of action. The Absentminded Driver Paradox lies in the fact that it may be worthwhile to reconsider and deviate (although no new information has been obtained). The question is, if this can happen.

Preliminary Remarks: (i) If the decision maker does not take the optimal decision rule from the *available decision rules*, it is common sense that he wishes to reconsider, if he can, and feel sorry, if he cannot. This emphasizes, once again, the importance of the feasibility problem.

(ii) Furthermore, the wish to deviate halfway seems to be natural, as the decision maker wants to imitate the perfect information-optimal strategy $H^p * E$ as close as possible, given his absentmindedness.

Aumann et al. (1997^a) mainly consider the question if optimal behavior strategies are time consistent. They consider one kind of deviations and do not pose (or answer) the question if a ‘reconsidering strategy’ could be better. Gilboa (1997) transforms the P & R-example into a 2-agent strategic game with imperfect information and identical payoff functions and claims it to be an equivalent presentation of the problem. Lipman (1997) thinks that both approaches are equivalent. *We do not agree* (see section 5, point (iii)).

3. Analysis of the absentminded driver.

Let us analyze the decision maker’s problem at the stochastically chosen time moment τ .

According to good traditions of decision theory this decision problem can be split up into two steps:

- (i) Assign (by reasoning about the past: forward induction) probabilities

$$\alpha_1, \alpha_2, \dots, \alpha_n \text{ to the events } E(C_1), E(C_2), \dots, E(C_n),$$

being at time τ in C_1, C_2, \dots, C_n , respectively.

- (ii) Choose (by reasoning about the future: backward induction) the most profitable action (H or E) i.e. the action that gives the highest expected payoff.

As we shall see, both steps require additional assumptions about the decision makers’ memory and abilities. P & R (see Piccione and Rubinstein (1997^b)) refuse to make additional assumptions but in the reactions several attempts are made to fill the holes in the model.

(i) Assessing probabilities

Let $\pi_i, i = 1, \dots, n$ be the probability that $\tau = i$. Note that $\pi_1 + \dots + \pi_n$ need not be 1; an additional option could be $\tau = n + 1$ (too late to reconsider). If the behavior rule used before τ is given by $[p : 1 - p]$ (and the probabilities π_j and p are implemented by independent chance mechanisms), the chance to be at crossing C_j at the moment τ is $\pi_j p^{j-1}$. The probability to be at the terminal node Z_j is $\left[\sum_{k>j} \pi_k \right] p^{j-1} (1 - p)$.

Therefore, the probability α_j to be at the crossing C_j , *conditional to the fact that up to the moment τ the rule $[p : 1 - p]^{\tau-1}$ has been followed*, is

$$(\pi_1, \pi_2 p, \dots, \pi_n p^{n-1}).$$

The numbers α_j are not ‘beliefs’ but objective probabilities, in as far as the probabilities π_j are objective probabilities (results of an independent chance mechanism). Piccione and Rubinstein (1997) calls the numbers $\alpha = \alpha_1$ and $1 - \alpha = \alpha_2$ subjective probabilities (beliefs), no value is excluded in advance and no reason is given why the driver has these beliefs. Probabilities, derived in the way we did, are called *consistent beliefs*. Aumann, Hart and Perry (1997) argue (and we think they are right) that only consistent beliefs are *well-founded* and that only the values $\pi_1 = \dots = \pi_n$ do fully right to the absentmindedness assumption. They also find probabilities with the same proportions as we did. In the P & R example they find $\alpha = \frac{1}{1+p}$ because they apply Bayesian updating.⁶

In the remaining part of the paper we assume that $\pi_i = \frac{\pi}{n}$ for all i with $0 \leq \pi \leq 1$. Mostly we take $\pi = 1$.

Conclusion: *If the decision maker knows the probabilities $\pi_j = \frac{\pi}{n}$, $j = 1, \dots, n$ (the probabilities that takes the absentmindedness seriously) and remembers his pre-meditated action $[p : 1 - p]$, he must assess the probabilities $\alpha = \frac{\pi}{n}(1, p, \dots, p^{n-1})$ to the events ‘being in C_j at time τ ’.*

(ii) Choosing the optimal action

Suppose we have assessed the probabilities as above. Then we come to the second more complicated decision: the choice between the actions H and E. Here the future plays a role or, more precisely, what the decision maker expects from the future. The option E gives a lottery

$$[\alpha_1 : Z_1] \oplus [\alpha_2 : Z_2] \oplus \dots \oplus [\alpha_n : Z_n]$$

and the option H gives the lottery

$$[\alpha_1 : \bar{E}(C_2)] \oplus [\alpha_2 : \bar{E}(C_3)] \oplus \dots \oplus [\alpha_n : Z_{n+1}]$$

where $\bar{E}(C_j)$ is the event that the taxi is at crossing C_j at time $\tau + 1$, a moment he does not deviate anymore. The decision maker has to decide which of the lotteries he prefers. To be able to do so, the decision maker has to assess *reasonable* utility values U_j to the events $\bar{E}(C_j)$ for $2 \leq j \leq n$, being at moment $\tau + 1$ at crossing C_j .

Remark: Note that assessing the values of U_j is speculating about the future

⁶ Because of Bayesian updating a restant of the paradox remains in the analysis of AHP. They restore the time consistency but still the expected utility is higher in the action stage than in the planning stage. The discussion about ‘feeling sorry or not’ (pp. 108–109) is a consequence of it. At the end of Section 4 we will discuss this point.

but, whatever the result may be, it must satisfy the inequalities $\min_{k \geq j} u_k \leq U_j \leq \max_{k \geq j} u_k$, as **all** potential outcomes are lotteries over the outcomes Z_j, \dots, Z_{n+1} . In P & R only the value of $\bar{E}(C_2)$ has to be assessed and $u_3 = 1 \leq U_2 \leq 4 = u_2$. Then the choice is between a lottery with expected utility $2p$ and the lottery

$$[\frac{1}{2} : \bar{E}(C_2)] \oplus [\frac{p}{2} : Z_3].$$

Therefore, the move H is certainly the best choice if $p < \frac{1}{3}$ i.e., $\alpha = \alpha_1 > 0.75$. P & R state that E is the best action for all $\alpha < 1$ and equally good as H for $\alpha = 1$. This shows, once again, that P & R's analysis is mistaken (as Aumann et al. (1997^a) noticed already).

Time inconsistency will occur if (a) $p = 0$, (not surprising, as $p = 0$ is the worst strategy. So the decision maker is glad to get the opportunity to change his action), (b) $p = 1$ and $U_2 < 3$ and (c) $p \in (0, 1)$ and $U_2 \neq 3p$.

To say more about the values of U_j , $j \geq 2$ (i.e., to say more about the future) we have to stipulate in what way the decision maker can deviate from his initial strategy $[p : 1 - p]^{\tau-1}$. There are at least three possibilities:

(i) At time τ the decision maker is allowed to choose between (E) and (H) and in the future (if there is a future) the strategy $[p : 1 - p]^{n-\tau}$ is resumed. The decision maker takes the decision only at time τ in his own hands. This seems to be the case that is investigated in Aumann et al. (1997^a) and Gilboa (1997). With small inessential differences it is the *multiselves approach* of Piccione and Rubinstein (1997), the *modified multiself approach* of Battigalli (1997) and *modified multiself approach* of Halpern (1997).

(ii) At time τ the decision maker is allowed to supplement $[p : 1 - p]^{\tau-1}$ with $[r : 1 - r]^{n-\tau+1}$ for any $r \in [0, 1]$: he is allowed to change the odds of the randomizer. This is the *PR-time consistency* in Halpern (1997) and the *time consistency* of Piccione and Rubinstein.

(iii) At time τ the decision maker is allowed to choose between (E) and (H) **as well as** to change the odds of the randomizer.

The three possibilities can be memorized by the following self-explaining notation:

$$\begin{array}{ll} [p : 1 - p]^{\tau-1}, [q : 1 - q], [p : 1 - p]^{n-\tau} & \text{case (i)} \\ [p : 1 - p]^{\tau-1}, [r : 1 - r]^{n-\tau+1} & \text{case (ii)} \\ [p : 1 - p]^{\tau-1}, [q : 1 - q], [r : 1 - r]^{n-\tau} & \text{case (iii)} \end{array}$$

In the next section we repeat the analysis of the absentminded driver problem under the condition that action plans as mentioned in (i), (ii) and (iii) are **feasible**. We will

return to the feasibility question later on.

4. Analysis of the absentminded driver problem (continued).

For $k = 1, \dots, n+1$ we introduce the functions $F_k: [0, 1] \rightarrow \mathbf{R}$ defined by

$$F_k(t) := \sum_{j=0}^{n-k} t^j (1-t) u_{k+j} + t^{n-k} u_{n+1}.$$

The functions $F_k(t)$ is the expected utility, if we start in the point C_k and the behavior strategy $[t : 1-t]^{n-k}$ is followed. The function $F_1 =: F$ is to be maximized to find the optimal behavior strategy. We order the possible states in which the decision maker is $C_1, \dots, C_n, Z_{n+1} = M, Z_1, \dots, Z_n$. For $t \in [0, 1]$ the transition matrix of the behavior rule $[t : 1-t]$ is given by the stochastic $(2n+1) \times (2n+1)$ -matrix $X(t)$ (we deleted the node B):

	C_1	C_2	\dots	\dots	C_n	M	Z_1	Z_2	\dots	Z_{n-1}	Z_n
C_1	0	t	\dots			0	$(1-t)$		\dots		
C_2		0	t			0		$(1-t)$			
\vdots			\ddots	\ddots		\vdots			\ddots		
C_{n-1}				0	t	0				$(1-t)$	
C_n					0	t					$(1-t)$
M						1					
Z_1						0	1				
Z_2						0		1			
\vdots						\vdots			\ddots		
Z_{n-1}						0				1	
Z_n						0					1

The incumbent strategy followed up to the time moment τ , $[p : 1-p]^{\tau-1}$ yields the transition matrix $Y(\pi, p) := \frac{\pi}{n} \sum_{i=1}^n X(p)^{t-1} + (1-\pi) X(p)^n$.

By left multiplication of the matrix $Y(\pi, p)$ to the vector $e = (1, 0, 0, \dots, 0)$ we get the chance vector $e Y(\pi, p)$, giving the probabilities of being in the different states at time τ . It has the coordinates:

$$IP(C_i | \pi, p) = \frac{\pi}{n} p^{i-1}, \quad IP(Z_i | \pi, p) = (1 - \frac{\pi}{n}) p^{i-1} (1-p), \quad IP(M | \pi, p) = (1-\pi) p^n.$$

So, at the stochastic time moment τ we have a lottery

$$[\frac{\pi}{n} : E(C_1)] \oplus [\frac{\pi}{n} p : E(C_2)] \oplus \dots \oplus [\frac{\pi}{n} p^{n-1} : E(C_n)]$$

and an already fixed utility $\sum_{i=1}^n (1 - \frac{\pi}{n}) p^{i-1} (1-p) u_i + (1-\pi) p^n u_{n+1}$.

If we choose (in the cases (i) and (iii)) the option E we obtain the lottery

$$[\frac{\pi}{n} : Z_1] \oplus [\frac{\pi}{n} p : Z_2] \oplus \dots \oplus [\frac{\pi}{n} p^{n-1} : Z_n] \text{ with expected utility } \frac{\pi}{n} \sum_{i=1}^n p^{i-1} u_i.$$

The option H gives the lottery $[\frac{\pi}{n} : \bar{E}(C_2)] \oplus [\frac{\pi}{n} p : \bar{E}(C_3)] \oplus \dots \oplus [\frac{\pi}{n} p^{n-1} : M]$.

If we continue with $[r : 1-r]^{n-\tau}$, the expected value of $\bar{E}(C_j)$ is $F_j(r)$. In case (i) we take $r = p$. In case (ii) the expected value of $E(C_j)$ is $F_j(r)$ too.

After these preparations we can formulate the decision problems in the different cases:

Case (i): the decision at time τ is between (we write only the part of the problem

not yet settled and delete the factor $\frac{\pi}{n}$): $\sum_{i=1}^n p^{i-1} u_i$ and $\sum_{i=1}^n p^{i-1} F_{i+1}(p)$.

If $\sum_{i=1}^n p^{i-1} (F_{i+1}(p) - u_i) > 0$, the optimal choice is $q = 1$, the option H,

if $\sum_{i=1}^n p^{i-1} (F_{i+1}(p) - u_i) < 0$ the optimal choice is $q = 0$, the option E,

in case of equality any choice is optimal.

To show the relation between optimality and time consistency of a behavior strategy, we will prove:

Proposition 4. (Aumann et al. (1997^a)) *The polynomial $\sum_{i=1}^n p^{i-1} (F_{i+1}(p) - u_i)$ and the polynomial $F'(p)$ are the same.*

Proof: The derivative of $F(t) = \sum_{i=1}^n t^{i-1} (1-t) u_i + t^n u_{n+1}$ equals

$$-\sum_{i=1}^n t^{i-1} u_i + \sum_{i=2}^n (i-1) t^{i-2} (1-t) u_i + n t^{n-1} u_{n+1}.$$

We have to prove that the second and third term of this expression equals

$$\sum_{i=1}^n t^{i-1} F_{i+1}(t).$$

The coefficient of u_k , $k \leq n$ is the polynomial

$$\sum_{i < k} t^{i-1} t^{k-i-1} (1-t) = (k-1) t^{k-2} (1-t).$$

The coefficient of u_{n+1} is $\sum_{i=1}^n t^{i-1} t^{n-i} = n t^{n-1}$ ◀

So, by the proposition, the optimal choice is H, E or $\Delta(H, E)$ (any mixture of H and E) if $F'(p) > 0$, < 0 or $= 0$, respectively. So $p = 0$ is time consistent iff $F'(0) \leq 0$, $p \in (0, 1)$ is time consistent iff $F'(p) = 0$ and $p = 1$ is time consistent if $F'(1) \geq 0$.

This means that optimal behavior strategies are always time consistent but that even the worst behavior strategy is time consistent, if it is a stationary point of F . *For the decision maker time consistency is not always a virtue* (as it is under perfect recall, see Theorem 3).

In the P & R example the decision is between $4p$ and $[4(1-p) + p] + p = 4 - 2p$. If we take $\pi = 1$, the total expected payoff (the fixed part is zero, as $u_1 = 0$) is $\frac{1}{2} \max(4p, 4 - 2p)$. We see that $p = \frac{2}{3}$ is the unique time consistent strategy but also that $(p, q) = (0, 1)$ and $(1, 0)$ are (time inconsistent) strategies with the highest expected payoff 2. In the set of actions of type (i) these strategies are the best imitation of the perfect information-optimal strategy $H^2 * E$. One may be surprised that the worst behavior strategy $p = 0$ can become the best, if this kind of deviation is allowed. The following may explain this point. The decision maker ‘hopes’ that the strategy $p = 0$ will never be used and that he can reconsider at $\tau = 1$. After reconsidering the $(p = 0)$ -strategy is resumed and now it has become the best action. So his only ‘fear’ is that $\tau = 2$.

In AHP we must compare $7 + 22p^3$ with $(22p - 20p^2) + p(22 - 20p) + 2p^2 = 44p - 38p^2$. Time consistency occurs iff $p = 0$, $p = \frac{7}{30}$ and $p = \frac{1}{2}$, the same numbers as in AHP. The total expected payoff is now, if we take $\pi = 1$:

$$7 * \frac{2}{3}(1-p) + 0 * \frac{1}{3}p(1-p) + \frac{1}{3}(7 + 22p^2) + \frac{1}{3}[F'(p)]_+$$

where $F'(p) = -7 + 44p - 60p^2$ and $a_+ = \max(a, 0)$. To determine the optimal choice for p we have to maximize the function $p \rightarrow 7 - \frac{14}{3}p + \frac{22}{3}p^2 + \frac{2}{3}p^3 + \frac{1}{3}[F'(p)]_+$. Here we stop the analysis because the computation in case (iii) will solve the problem in case (i) and (iii) as well.

Conclusion *We find the same time consistent behavior strategies as Aumann et al. (1997^a) but by a different reasoning. The optimal behavior strategies are time consistent but there may be more time consistent behavior rules. Moreover, time inconsistent decision rules often give better results.*

Case (ii): We consider the part of the decision problem, not yet settled, and

$$\text{maximize the function } r \rightarrow \frac{\pi}{n} \sum_{i=1}^n p^{i-1} F_i(r) \text{ over } r \in [0, 1].$$

The result is, in general, a *correspondence* (multivalued function) $p \rightarrow R(p)$. $r \in R(p)$ means that r is the best *continuation* of p . If $p \in R(p)$ the behavior strategy p is time consistent. If we apply this in P & R, the function $r \rightarrow \frac{1}{2}[(4r - 3r^2) + p(4 - 3r)]$ is to be maximized and we find $R(p) = \frac{2}{3} - \frac{1}{2}p$. The time consistent value for p is the solution of the equation $p = \frac{2}{3} - \frac{1}{2}p$ i.e., $p = \frac{4}{9}$. Here the number $1 - p = \frac{5}{9}$ (in

Piccione and Rubinstein) comes show up. The optimal time inconsistent strategy, the behavior strategy p that maximizes $p \rightarrow \frac{1}{2} [(4r - 3r^2) + p(4 - 3r)]_{r=R(p)}$, is $p = 1$ and $r = \frac{1}{6}$. The maximal utility is $\frac{49}{24}$, slightly more than 2.

Conclusion *The mysterious number $p = \frac{4}{9}$ appears here as the unique time consistent behavior rule. In particular, an optimal behavior rule is not necessarily time consistent, if this kind of deviations is considered.*

Case (iii) In this case we compare the functions

$$p \rightarrow \sum_{i=1}^n p^{i-1} u_i \text{ with } p \rightarrow \max_{r \in [0,1]} \left[\sum_{i=1}^n p^{i-1} F_{i+1}(r) \right]$$

to find the optimal value for q . Then we have to maximize over $p \in [0, 1]$ to find the optimal (time inconsistent) strategy. Time consistency occurs if $p = q = r$.

For P & R we find the following problems:

Maximize $r \rightarrow (4 - 3r) + p$: the result is $r = 0$.

Compare $4p$ with $4 + p$: the result is $q = 1$

and finally, maximize over p the expression $\frac{1}{2}(4 + p)$: the result is $p = 1$ and the maximal utility is $\frac{5}{2}$.

This strategy is the best imitation of the perfect information-optimal strategy $H^2 * E$.

In case $\tau = 1$ the payoff is 4 and in case $\tau = 2$ it is 1.

In AHP we find the following series of problems:

Maximize $r \rightarrow (22r(1-r) + 2r^2 + p(22 - 20r) + 2p^2)$: the solution is $R(p) = \frac{11}{20} - \frac{1}{2}p$.

Compare $7 + 22p^2$ with $[(22r - 20r^2) + p(22 - 20r) + 2p^2]_{r=R(p)}$:

for $p = \frac{1}{10}$ and $p = \frac{19}{30}$ H and E are equally good. This give $r = \frac{1}{2}$ and $r = \frac{7}{30}$ and no time consistency.

For $p \in [\frac{1}{10}, \frac{19}{30}]$ H is optimal and for the other values of p E is optimal.

For $p = 1$ followed by $q = 0$ and $r = \frac{1}{20}$ we find an optimal strategy. The value for r does not matter but the value $r = \frac{1}{20}$ has the flavor of a ‘trembling hand perfectness’: if $q = \varepsilon > 0$ is chosen by mistake, it is better to choose $r = \frac{1}{20}$. The optimal time inconsistent expected utility is $\frac{29}{3}$. By choosing $r = q = 0$ or $r = p$ we get the same expected utility but now by a deviations of type (ii) and (i), respectively.

Conclusion *Again time inconsistent behavior is better than optimal behavior strategies and optimal behavior strategies are not time consistent. Not so surprising, time inconsistent plans of this kind give the best result. This kind of strategies form the set of the most versatile strategies of the cases (i), (ii) and (iii).*

In the example AHP there are, in case (i), three time consistent values for p , namely

$p = 0$, $p = \frac{7}{30}$ and $p = \frac{1}{2}$. In the paper of Aumann et al. there is a discussion that the decision maker playing the optimal strategy $p = 0$ ‘feel sorry (at time τ) not to have played $p = \frac{1}{2}$ ’ (pp. 108–109). Let us analyze the situation. At time τ we have the following probabilities (see table):

	C_1	C_2	C_3	Z_1	Z_2	Fixed utility	+ Utility from ‘exit’
$p = 0$	$\frac{1}{3}$	0	0	$\frac{2}{3}$	0	$\frac{14}{3}$	$+\frac{7}{3} = 7$
$p = \frac{7}{30}$	$\frac{1}{3}$	$\frac{7}{90}$	$\frac{49}{2700}$	$\frac{46}{90}$	$\frac{161}{2700}$	$\frac{322}{90}$	$+\frac{3689}{1350} = 6.31$
$p = \frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{7}{3}$	$+\frac{25}{6} = 6.5$

Playing ‘exit’ (this is easier to compute but playing ‘continue’ gives the same expected payoff) for $p = \frac{7}{30}$ or $p = \frac{1}{2}$ gives a utility of $\frac{7}{3} + \frac{1078}{2700} = \frac{3689}{1350}$ and $\frac{7}{3} + \frac{11}{6} = \frac{25}{6}$, respectively. There is no reason to feel sorry not to have played one of these strategies (see the total utility in the tabel). Aumann et al. come to their result because they update the not-yet-settled utility. Then the last strategy gives $\frac{12}{7} * \frac{25}{6} = \frac{50}{7} > 7$.

We conclude this section with a discussion of Gilboa (1997). Gilboa has an other incentive to reconsider the absentminded driver story: he seeks to avoid absentmindedness. He formulates an alternative story he claims to be equivalent with the absentminded driver problem. The decision maker in his setting is represented by two agents. To keep the idea ‘absentmindedness’ he takes three precautions: (i) the agents are “twins” or more precisely “identical twins” (ii) with equal probability each of them is called to act first and finally (iii) they have to choose the same action (as twins and certainly identical twins are supposed to do). In the description of the “twins” Gilboa enters the psychopathology. He writes:

Each of them thinks of herself, when called for a decision, as “the self” and of her twin as “the other”. she has a way to define herself and thus she considers the twin’s decisions as independent of hers..... the two “versions” of the agent cannot uniquely define themselves, ... each agent can define herself at least at the time of her decision (which is the time of her glory)

It is hard to see why Gilboa takes all these precautions. Let us assume, the decision maker takes two agents Mary and John, no twins, not of the same gender and without any identity problem. Mary acts in information set X_{Mary} and John in information

set X_{John} . Each of them gets his instructions from the decision maker and a chance move determines who is acting first and who is acting second. The two agents do not know the order in which they are called for action as the decision tree shows (see figure 6).

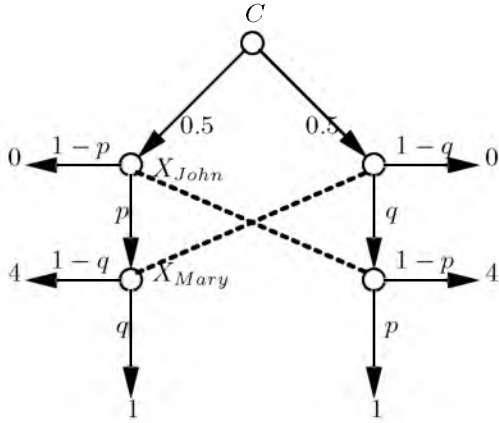


Figure 6: Gilboa's game.

Let us analyze the game. The two person game has three Nash equilibria namely $p = q = \frac{2}{3}$, $p = 1$ and $q = 0$ and $p = 0$ and $q = 1$ (p is John's action and q Mary's action). The first equilibrium is symmetric and gives the payoff $\frac{4}{3}$ and the other two equilibria give a payoff 2. We recognize the strategies and the payoffs: $\frac{4}{3}$ is the optimal behavior strategy payoff and 2 the payoff after an optimal deviation of type (i).⁷ In the more generalized situation we think that the number of 'crossing' information sets and the number of agents will increase dramatically. So we cannot see the advantage of Gilboa's approach.

5. Conclusions

(i) The absentminded driver paradox exists in the sense that reconsidering and deviating at a stochastic time moment gives a better result. Phrased in this way it is no longer a paradox: the best way to imitate the perfect information-optimal decision rule $H^p * E$ is to change your mind. To keep the idea of absentmindedness the decision

⁷ The strategies (E,H) and (H,E) are the usual strategies for twins: one behaves well and the other behaves badly and they cannot be punished because the outside world has an 'identity problem.'

maker must be able to choose a time moment stochastically. It is the question if this is possible. (see (vii)).

(ii) As already noticed by Aumann et al. (1997^a), the analysis of Piccione and Rubinstein is flawed. According to our analysis the following should be changed:

(a) The events $(C_1, t = 1)$ has probability $\frac{1}{2}$, the event $(C_2, t = 2)$ has probability $\frac{p}{2}$ and $(Z_1, t = 2)$ has probability $\frac{1-p}{2}$, if the incumbent strategy is p .

(b) It is not wise to make a ‘Bayesian update’ of these probabilities and to compare the unconditional maximal expected payoff $\frac{4}{3}$ with conditional expected payoffs. By doing so, the worst event $(Z_1, t = 2)$ is ignored. No wonder that the expected payoff is higher.

(c) Even if updated beliefs are considered, it is not true that, for all $\alpha < 1$, ‘exit’ is the best action (see Section 3).

(iii) The observation of Lipman (1997) that Gilboa and Aumann et al. give the same analysis is not right. In Aumann et al. the variables p and q have an asymmetric *interpretation* (the variable q , e.g., is the behavior rule used at *all other* intersections, it is the ‘incumbent strategy’ and it does not matter that in the P & R-example there is only one other intersection.) It is, therefore, not surprising that the variables p and q occur asymmetrically in their function $h(p, q)$. In as far as Aumann et al. only investigate time consistency, this formula is fine and it covers our case (i). In Gilboa’s analysis the variables have a symmetric interpretation and occur symmetrically. We have the feeling that Gilboa’s 2-agent game is not an alternative formulation of the *same* decision problem. Moreover, the number of agents increases dramatically, when n increases. By the way, the variables p , q and r in our analysis have clearly an asymmetric interpretation: p is the behavior rule before reconsidering, q at time of reconsidering and r thereafter.

(iv) In case (ii) the (at least for us) mysterious time consistent value $\frac{4}{9}$ comes show up, as the unique time consistent value for p in case (ii).

(v) The cases (i) and (ii) in the present paper are more or less equivalent with multiselves consistency and PR-consistency in Grove and Halpern (1997). ‘More or less’ because Grove and Halpern choose at random a node $x \in X$ where the decision maker reconsiders and we take a stochastic time moment.

(vi) It is confusing that most of the authors delete the node B and the move from B to C_1 . The decision maker is ‘plunged into oblivion immediately’ and has no idea how to

assess probabilities; everything is possible. If you do not know where you are, it is not a sound basis for decision making to assume that you are at any place with the same probability. So, here we have clearly a different problem. It should be noticed that also Piccione and Rubinstein give a picture (Example 1) that is not in accordance with their story.

(vii) The question of feasibility is the last problem to reconsider. ‘Is it sufficient to keep the idea of absentmindedness by saying that the value of τ is *stochastic* with *equal* probabilities for the values of $\tau \leq n$, where reconsidering matters?’ And ‘how can the choice of τ be implemented?’

A suggestion for an answer to the latter question could be found in an n -possibility variant of ‘tying knots’ in Aumann et al. i.e., a circular device with n positions (like a clock). At one position an alarm goes but the decision maker does not know this position (otherwise he has an alarm clock to cure his absentmindedness). So, he puts the indicator at a position and puts it one step farther at each crossing. If the alarm goes, he reconsiders and deviates, if necessary, from the behavior strategy as in the cases (i), (ii) or (iii). Note that the choice for q and r can also be pre-meditated in the ‘bar’. This device is sufficient to implement each of the variants (i), (ii) and (iii). In case (i) the decision maker takes the decision at time τ in his own hands, in case (ii) he pushes a button that puts the odds in the randomizer from p to r and in case (iii) he does both. We fear, however, that this device comes terribly close to an alarm clock. If the actions (i), (ii) and (iii) (or more complicated actions, like reconsidering at several stochastic time moments) are judged *infeasible*, the absentminded driver paradox disappears! It is curious to see that Aumann et al. reject the idea of an external device that chooses $\tau = 1$ and $\tau = 2$ with equal chances but use the idea to derive the formula for $h(p, q)$.

(viii) How sensitive the situation is for adding some external devices or processes can be seen from the following strategy (that looks quite feasible): in the planning stage the decision maker prepares $n + 1$ behavior rules p_0, p_1 up to p_n and a chance mechanism $t : (1 - t)$. At each crossing he takes the next behavior rule with probability t and uses the old behavior rule with probability $(1 - t)$. He does not need to know which behavior rule he is using. For $t = 0$ the behavior strategy p_0 is used all the time but for $t = 1$ the decision maker has with his strategy an built-in alarm clock if the k -th exit is the right exit and he takes $p_i = 1$ for $i = 0, \dots, k - 1$ and $p_k = 0$.

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